Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Stephan Dahlke, Filippo De Mari
Philipp Grohs, Demetrio Labate, Editors

Harmonic and Applied Analysis

From Groups to Signals

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Applied and Numerical Harmonic Analysis

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From Groups to Signals

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlyingmathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptivemodeling inherent in time-frequency-scalemethods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

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John J. Benedetto
Series Editor

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Contents

1 From Group Representations to Signal Analysis 1
Stephan Dahlke, Filippo De Mari, Philipp Grohs, and Demetrio Labate
References 5

2 The Use of Representations in Applied Harmonic Analysis 7
Filippo De Mari and Ernesto De Vito
2.1 Representations of Lie Groups 7
2.1.1 Locally compact groups 8
2.1.2 Lie Groups and Lie Algebras 10
2.1.3 Representation Theory 30
2.1.4 Reproducing Systems and Square Integrability 36
2.1.5 Unbounded Operators 43
2.1.6 Stone's Theorem and the Differential of a Representation ... 46
2.2 The Heisenberg Group and its Representations 48
2.2.1 The Group and its Lie Algebra 48
2.2.2 Automorphisms 51
2.2.3 The Schrödinger Representation 53
2.2.4 Time-frequency Analysis 57
2.3 The Metaplectic Representation 61
2.3.1 More on the Symplectic Group 62
2.3.2 Construction of the Metaplectic Representation 64
2.3.3 Restriction to Triangular Subgroups 67
References 80

3 Shearlet Coorbit Theory 83
Stephan Dahlke, Sören Häuser, Gabriele Steidl, and Gerd Teschke
3.1 Introduction 83
3.2 Coorbit Space Theory 85
3.2.1 Coorbit Spaces 85
3.2.2 Discretization 95

3.2.3 Examples 104
3.2.4 Proof of Lemma 3.16 and 3.18 108
3.3 Multivariate Shearlet Transform 110
3.3.1 The Shearlet Group in $\mathbb{R}^d$ 110
3.3.2 The Continuous Shearlet Transform 115
3.4 Multivariate Shearlet Coorbit Theory 119
3.4.1 Shearlet Coorbit Spaces 119
3.4.2 Discretization 127
3.5 Structure of Shearlet Coorbit Spaces 129
3.5.1 Density 129
3.5.2 Embeddings 130
3.5.3 Traces of Shearlet Coorbit Spaces 136
3.6 Variation of a Theme: the Toeplitz Shearlet Transform 145
References 146

4 Efficient Analysis and Detection of Edges Through Directional Multiscale Representations 149
Kanghui Guo and Demetrio Labate
4.1 Introduction 149
4.2 The continuous wavelet transform and its generalizations 153
4.2.1 Continuous wavelets 153
4.2.2 Continuous shearlets 157
4.2.3 Continuous shearlets in the plane ($n = 2$) 158
4.3 Shearlet analysis of step edges. Case $n = 2$. 163
4.3.1 Proof of Theorem 4.4 167
4.4 Shearlet analysis of edges in dimension $n = 3$ 175
4.4.1 3D Continuous Shearlet Transform 175
4.4.2 Characterization of 3D Boundaries 177
4.4.3 Identification of curve singularities on the piecewise smooth surface boundary of a solid 180
4.5 Other results and applications 187
4.5.1 Shearlet analysis of general edges 187
4.5.2 Numerical applications 188
Appendix 189
References 195

5 Optimally Sparse Data Representations 199
Philipp Grohs
5.1 Introduction 199
5.1.1 Notation 202
5.2 Signal Classes and Encoding 202
5.3 Upper Bounds on the Optimal Encoding Rate 207
5.4 Sparse Approximation in Dictionaries 214
5.4.1 Best N-term Approximation in Dictionaries 215

5.4.2 Best N-term Approximation with Polynomial Depth Search . 217
5.4.3 Sparse Approximations in Frames ... 220
5.5 Wavelet Approximation of Piecewise Smooth Functions ... 225
5.6 Cartoon Approximation with Curvelet Tight Frames ... 229
5.6.1 Suboptimality of Wavelets for Cartoon Images ... 229
5.6.2 Curvelets ... 233
5.6.3 Shearlets ... 240
5.7 Further Examples ... 244
5.8 Appendix: Chernoff Bounds ... 245
References ... 247

Index ... 249

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Chapter 1
From Group Representations to Signal Analysis

Stephan Dahlke, Filippo De Mari, Philipp Grohs, and Demetrio Labate

Abstract In this chapter, we present the point of view that has inspired this book and we explain the perspective and scope of the four chapters that follow.

This book is concerned with signal analysis in its broadest sense. Usually, signals are modeled as functions in suitable spaces such as L^2, the space of square integrable functions, or Sobolev spaces. Signals might be given explicitly as, for example, in image analysis or implicitly, as solutions of operator equations. In either case, the problem of interest is to analyze and process these signals, that is, to extract their information and then to manipulate the signals for tasks such as compression, denoising, and enhancement.

During the last decade, rapid advances in computing power and sensing technologies, and the exponential growth of the internet have enormously increased the availability of data, leading to what is sometimes described as the "data deluge" or "big data" problem. This situation created new opportunities and new challenges in the field of signal processing, since huge amounts of data have to be transmitted, stored, and analyzed with high efficiency. The challenges are due not only to the size of data but also to their complexity, since data acquired in many applications (think, for instance, of electronic surveillance and social media data) are often

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heterogeneous and high-dimensional. Confronted and perhaps even daunted by these challenges, some scientists have already announced the “end of theory.” They claim that a rigorous mathematical theory is no longer necessary as big data “offer a higher form of intelligence and knowledge that can generate insights that were previously impossible” [1]. The whole analysis process should hence be solely data-driven because it is claimed that “correlation is more important than causation.” In other words, knowledge would be generated by combing through data with sufficient computational power in order to discover correlations by means of appropriate statistical tools. According to this point of view, the future progress of science would only depend on the increase of computing power and the clever implementation of well-established statistical algorithms.

We are guided by a very different perspective. We believe that there is nothing more applicable than a sound mathematical theory. We are convinced that rigorous mathematics can provide, and is already providing, the right instruments to address the many new challenges coming from advances in science and technology. Harmonic analysis, in particular, has been historically a discipline at the crossroads of mathematics, computer science, and engineering, and because of its versatile nature it can boost a closer mutual cooperation, where mathematical theory fosters technological advances and scientific discoveries, and, in turn, science and technology provide a continuous stimulus for the development of new mathematics. Starting with classical Fourier analysis and continuing with the theory of time-frequency analysis and wavelets, harmonic analysis has shown over many decades a remarkable “regenerative and centralizing power”[1]. During the last decade, the emergence of the theory of sparse representations and compressed sensing and the introduction of innovative multiscale methods going beyond conventional wavelets prove that we are witnessing a clear manifestation of our point of view. The aim of this book is to present in a comprehensive and consistent manner some of the most promising mathematical concepts emerged in applied harmonic analysis during the last decade.

Let us briefly summarize these ideas while emphasizing their connection with signal analysis.

The first step in signal analysis is always signal transformation. Signals are modeled as elements of a function space and transformed via a mapping into a new function, defined on a suitable parameter space. This mapping usually makes it easier to recognize and extract the most relevant information of the signal.

Very often the parameter space is itself highly structured, reflecting the symmetries inherently attached to the space of signals. Under favorable circumstances, the parameter space forms a group. Thus, the very high-powered tools from group theory apply, in particular the theory of unitary representations of Lie groups. In fact some of the most important transforms, such as the wavelet transform and the Gabor transform, are related to square-integrable group representations. The overall

[1] J. Benedetto, introduction to [2].

potential of representation theory, however, is far from having been fully exploited yet and, more importantly, its most basic constructs and techniques are not as widely known among applied mathematicians and people working in this area.

Thus, in the chapter "The use of representation theory in applied harmonic analysis" we provide a crash course on Lie theory, survey its most commonly recognized applications in signal analysis already indicated above, namely the representation theory of the affine group and of the Heisenberg group as keys to wavelets and Gabor analysis, respectively, and finally examine in some detail the metaplectic representation of the symplectic group. Even though the latter is also widely appreciated and its role clearly acknowledged in this and other closely related fields, some of its more recently investigated facets are perhaps less known. In particular, we focus on the observation that its restriction to block-triangular subgroups provides a unified approach to many known reproducing formulae, such as those used in the theory of shearlets, as well as some other new intriguing constructions emerged in applied harmonic analysis and signal processing.

One of the most successful applications of unitary group representations is coorbit theory, initially developed by Feichtinger and Gröchenig in a series of papers in the late eighties. It is well known that the convergence order of any numerical approximation scheme is closely related to the regularity of the object one wants to approximate. Therefore, it is important to classify the "right" smoothness measures. This is one of the key issues addressed by coorbit space theory, in which a crucial role is played by the so-called *voice transform*. The main data with which the transform is built are a unitary representation of the parameter group G, acting on the Hilbert space $\mathscr{H}$ of the signals that one wants to analyze, and a fixed vector in $\mathscr{H}$, to be thought of as the appropriate analogue of a standard wavelet. The voice transform of a signal is just the collection of its projections along the various "directions" in $\mathscr{H}$ that are obtained by acting with G on the analyzing vector, and thus maps signals in $\mathscr{H}$ into functions on G. The technical assumptions are taylored in such a way that the voice transform of any signal actually belongs to $L^2(G)$, and, most importantly, gives rise to a natural class of distributions, to which the (extended) voice transform may be applied. The idea is then to define canonical smoothness spaces, the coorbit spaces, by collecting those distributions whose voice transform has a prescribed decay rate, codified by the appropriate space of functions on G. In the wavelet setting, the coorbit spaces are the (homogeneous) Besov spaces, and for the Gabor transform one obtain the modulation spaces. In Section 3.2 we present a detailed introduction into coorbit space theory.

The next step in signal analysis is always discretization. When it comes to practical applications, of course only discrete data can be handled, so that some kinds of bases of more general frames for the underlying spaces are needed. The resulting building blocks, the frame elements, should therefore be adapted to the kind of information that one wants to extract. Typical examples are wavelet frames that can be used for time-scale analysis of signals, and Gabor frames that can be used for time-frequency analysis. One way to construct canonical building blocks is again coorbit theory. We refer again to Section 3.2.

One specific task in modern signal analysis is the detection of directional information. Classical transforms such as the wavelet and the Gabor transform, respectively, are suboptimal since they are essentially isotropic. Therefore, there has been a compelling need to design new transforms and building blocks that are particularly tuned to this problem. The main issue is the search of the correct mathematical tool to capture and follow directional changes, and this is achieved by using a most natural idea, the use of shearing transformations in combination with anisotropic (typically parabolic) dilations. Roughly speaking, this amounts to elongating shapes along one direction while keeping the others unchanged. Due perhaps to their intrinsic features, among other approaches such as curvelets, ridgelets, and contourlets, just to name a few, shearlets have gained more and more attention over the last few years and have proved to be very efficient. A discussion of some of the most significant aspects of shearlet theory covers a very important part of this book. Among all the recently developed directional transforms, the shearlet transform stands out since it stems from a square-integrable representation of a specific group, the full shearlet group. This remarkable property paves the way for the application of all the powerful tools from group representation theory already mentioned above and, in particular, square-integrability calls for an understanding of the coorbit space theory that naturally arises. In the third chapter of this book, “Shearlet coorbit theory,” the use of the continuous shearlet transform in the context of coorbit space theory is thoroughly investigated, and various embeddings and trace results are proved.

Once suitable building blocks for the extraction of directional information are constructed, a natural question is how we can benefit from it in terms of data compression. A thorough mathematical analysis of this question in a more general context is presented in the chapter “Optimally Sparse Data Representations” where we study the use of sparse frame representations for the optimal compression of different signal classes governed by different features (point- or curve-like discontinuities, textures, etc.). Several examples are considered, among them the optimal compression of piecewise smooth univariate signals with wavelets and the optimal compression of bivariate piecewise smooth functions with curved discontinuities (a common benchmark model for natural images known as ‘cartoon-images’) by curvelets and shearlets.

So far, we have mainly discussed group representations, their associated function spaces and their atomic decompositions, with special attention to building blocks that are sensitive to directions. In a certain sense, such function spaces are defined so that membership in these spaces is a measure of some kind of *global* information. However, in many practical applications, it is equally important or even preferable to measure *local* geometric information, e.g., local smoothness or local geometric features of singularity structures. The prominence of local properties, that was already critical in the chapter “Optimally sparse data representations” in the context of sparse approximations, is the central topic of the chapter “Efficient analysis and detection of edges through directional multiscale representations.” The main goal of this chapter is to illustrate how local information can be retrieved by means of microlocal analysis based on the continuous shearlet transform.